

Non-linear Brane Dynamics in Six Dimensions

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We consider a dynamical brane world in a six dimensional spacetime containing a singularity. Using the Israel conditions we study the motion of a 4-brane embedded in this setup. We analyze the brane behavior when its position is perturbed about a fixed point and solve the full non-linear dynamics in the several possible scenarios. We also investigate the possible gravitational shortcuts and calculate the delay between graviton and photon signals and the ratio of the corresponding subtended horizons.

1. The Brane Cosmological Model

We consider a six-dimensional model described by the following metric

$$ds^2 = -n^2(t, y, z)dt^2 + a^2(t, y, z)d\Sigma_k^2 + b^2(t, y, z)dy^2 + c^2(t, y, z)dz^2, \quad (1)$$

where $d\Sigma_k^2$ represents the metric of the three dimensional spatial sections with $k = -1, 0, 1$ corresponding to a hyperbolic, a flat and an elliptic space, respectively.

The matter content on the brane is directly related to the jump of the extrinsic curvature tensor across the brane [1,2]. This relation has been derived in the case of a static brane in a previous work [3]. Here we generalize our result for the Israel conditions to include the case of a brane moving with respect to the coordinate system, which position at any bulk time t is denoted by $z = \mathcal{R}(t)$.

The extrinsic curvature tensor on the brane is given by

$$K_{MN} = \eta_M^L \nabla_L \tilde{n}_N, \quad (2)$$

where \tilde{n}^A is a unit vector field normal to the brane worldsheet

$$\tilde{n}^A = \left\{ \frac{c\dot{\mathcal{R}}}{n^2\sqrt{1 - \frac{c^2}{n^2}\dot{\mathcal{R}}^2}}, 0, 0, 0, 0, \frac{1}{c\sqrt{1 - \frac{c^2}{n^2}\dot{\mathcal{R}}^2}} \right\}, \quad (3)$$

*I would like to thank Elcio Abdalla for useful discussions and for reading the manuscript. This work has been supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Brazil.

and

$$\eta_{MN} = g_{MN} - \tilde{n}_M \tilde{n}_N \quad (4)$$

is the induced metric on the brane, from which we can obtain a relation between dt (the bulk time) and $d\tau$ (the brane time),

$$d\tau = n(t, \mathcal{R}(t)) \sqrt{1 - \frac{c^2(t, \mathcal{R}(t))}{n^2(t, \mathcal{R}(t))} \dot{\mathcal{R}}^2} dt \equiv n\gamma^{-1} dt, \quad (5)$$

where a dot means derivative with respect to the bulk time t .

1.1. The Israel Conditions

The energy-momentum tensor on the brane located at z_0 can be written as

$$T_{MN}^{(b)} = \frac{\delta(z - z_0)}{c} \{(\rho + p)u_M u_N + p\eta_{MN}\}. \quad (6)$$

We also define a tensor \hat{T}_{AB} as

$$\hat{T}_{AB} \equiv T_{AB} - \frac{1}{4}T\eta_{AB}. \quad (7)$$

The Israel junction conditions [4] are given by

$$[K_{\mu\nu}] = -\kappa_{(6)}^2 \hat{T}_{\mu\nu}, \quad (8)$$

where the brackets stand for the jump across the brane and $K_{\mu\nu} = e_\mu^A e_\nu^B K_{AB}$, where e_μ^A form a basis of the vector space tangent to the brane worldvolume. The left-hand side of (8) can be calculated taking into account the mirror symmetry across the brane.

At this point it is convenient to choose a specific bulk metric of the form (1) satisfying six dimensional Einstein equations. This is given by

$$ds^2 = -h(z)dt^2 + a^2(z)d\Sigma_k^2 + h^{-1}(z)dz^2, \quad (9)$$

where

$$a(z) = \frac{z}{l}, \quad (10)$$

$$d\Sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega_{(2)}^2 + (1 - kr^2)dy^2, \quad (11)$$

$$h(z) = k + \frac{z^2}{l^2} - \frac{M}{z^3} + \frac{Q^2}{z^6} \quad (12)$$

with $l^{-2} \propto -\Lambda$ (Λ being the cosmological constant) and M and Q^2 are constants. We should notice that y is a compactified coordinate.

This metric contains a singularity located at $z = 0$ and it is valid on the $z < \mathcal{R}(t)$ parts of surfaces of constant t , and its reflection, by the Z_2 orbifold symmetry, is valid on the $z > \mathcal{R}(t)$ parts. If $M = 0$ and $Q^2 = 0$, then (9) is simply the metric of de Sitter or Anti de Sitter spacetime according to the sign of l^2 .

With this Ansatz the Israel conditions (8) reduce to only two equations, which read

$$\begin{aligned} \ddot{\mathcal{R}} + \frac{1}{2} \frac{h'}{h^3} \dot{\mathcal{R}}^4 - 3 \frac{h'}{h} \dot{\mathcal{R}}^2 + \frac{1}{2} h h' = \\ -\kappa_{(6)}^2 \left(\frac{3\rho + 4p}{8} \right) h^2 \left(1 - \frac{\dot{\mathcal{R}}^2}{h^2} \right)^{3/2} \end{aligned} \quad (13)$$

$$\frac{a'}{a} + \frac{\dot{\mathcal{R}}}{h^2} \frac{\dot{a}}{a} = \kappa_{(6)}^2 \frac{\rho}{8} \left(1 - \frac{\dot{\mathcal{R}}^2}{h^2} \right)^{1/2},$$

where all the metric coefficients must be evaluated on the brane. The system (13) describes the full non-linear dynamics of the brane embedded in the static bulk (9).

1.2. The Geodesic Equation and the Time Delay

We consider two points on the brane r_A and r_B . In general there are more than one null geodesics connecting them in the $1+5$ spacetime. The trajectories of photons must be on the brane and those of gravitons may be outside. The graviton path is defined equating (9) to zero. Since we are looking for a path that minimizes t when the final

point r_B is on the brane, the problem reduces to an Euler-Lagrange problem [5]. Then as in [6] the shortest graviton path is given by

$$\ddot{\mathcal{R}}_g + \left(\frac{1}{\mathcal{R}_g} - \frac{3}{2} \frac{h'}{h} \right) \dot{\mathcal{R}}_g^2 + \frac{1}{2} h h' - \frac{h^2}{\mathcal{R}_g} = 0. \quad (14)$$

The time delay of the photon traveling on the brane with respect to the gravitons traveling in the bulk measured by an observer on the brane can approximately be written as [6]

$$\begin{aligned} \Delta\tau \simeq \mathcal{R}(t_f) \int_0^{t_f} dt \left(\frac{1}{\mathcal{R}_g(t)} \sqrt{h(\mathcal{R}_g) - \frac{\dot{\mathcal{R}}_g(t)^2}{h(\mathcal{R}_g)}} - \right. \\ \left. - \frac{1}{\mathcal{R}(t)} \frac{d\tau}{dt} \right). \end{aligned} \quad (15)$$

It is also interesting to look at the ratio between the horizons subtended by the photons traveling on the brane and the gravitons traveling in the bulk,

$$\frac{g}{\gamma} = \frac{\int_0^{t_f} \frac{dt}{\mathcal{R}_g(t)} \sqrt{h(\mathcal{R}_g) - \frac{\dot{\mathcal{R}}_g(t)^2}{h(\mathcal{R}_g)}}}{\int_0^{t_f} \frac{dt}{\mathcal{R}(t)} \frac{d\tau}{dt}}. \quad (16)$$

2. Non-linear Brane Dynamics

The system (13) describing the brane dynamics can be numerically solved for several combinations of M , Q^2 , k and l^2 . When M , Q^2 and k vanish, the solution for $\mathcal{R}(t)$ is a constant or a linear function in t depending on the given initial condition for $\dot{\mathcal{R}}$.

For M and Q^2 non-zero we have solved (13) in the typical cases of a domain wall ($\omega = -1$), matter ($\omega = 0$) and radiation ($\omega = 1/3$) dominated branes. In Fig.1 we show all the possible forms of $h(z)$ due to M , Q^2 , k and l^2 combinations, which we study in this section. Some of our results are also illustrated in Fig. 2 together with the solution of the geodesic equation (14) in order to verify the possibility of having shortcuts. We have also calculated the time delays and the ratio between graviton and photon horizons for the examples of shortcuts appearing in Fig. 2, these are shown in Table 3 together with the graviton bulk flight time and its corresponding brane time according to the equation (5).

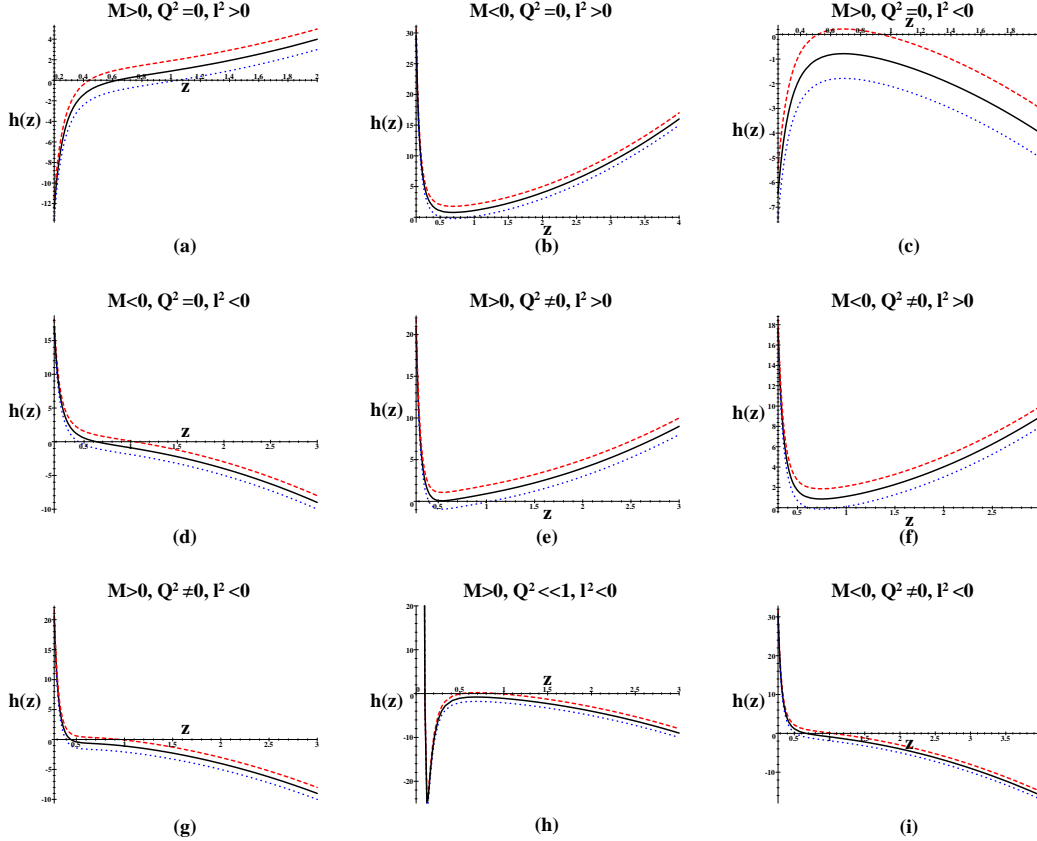


Figure 1. $h(z)$ for all the possible combinations of M , Q^2 and l^2 , including $k = 1$ (dashed lines), $k = 0$ (solid lines) and $k = -1$ (dotted lines) cases.

We have classified all cases according to the sign of the M parameter and to whether we are in dS or AdS bulks. Moreover, we studied the zero charge black hole as well as the Reissner-Nordström-type solutions, namely 8 cases. In Tables 1 and 2 it is displayed the behavior of the solutions of the brane equation of motion (13) for a domain wall, a matter, and a radiation dominated branes. We also show the geodesic behavior and remark the cases in which shortcuts are possible.

3. Discussion and Conclusions

In the present work we have studied the behavior of a brane embedded in a six dimensional de Sitter or Anti de Sitter spacetime containing a singularity. The system of equations describing this behavior from the point of view of an observer in the bulk appears to be highly non-linear.

By solving the full non-linear system we found different behaviors for the several scenarios appearing due to all the combinations of M , Q^2 , k and l^2 taken into account. The results show

Table 1

Scale factor and geodesics evolution (uncharged case). The arrow indicates the behavior tendency. There can be more than one behavior depending on the initial conditions. The star or the dagger in the last column indicates the possibility of shortcuts for the matter and radiation dominated branes or the domain wall, respectively.

M	Q^2	k	l^2	$h(R)$	DW	MDB	RDB	Geodesic
+	0	1	+	AdS-Schwarzschild	$\rightarrow r_H/\text{grow}$	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H^*/\text{grow}$
+	0	0,-1	+	AdS-topological black hole	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H^*$
-	0	0,1	+	AdS-naked singularity	grow	$\rightarrow 0$	$\rightarrow 0$	grow $^*/\rightarrow 0$
-	0	-1	+	AdS-topological black hole	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H$
+	0	1	-	dS-Schwarzschild	$\rightarrow r_H/\rightarrow r_c$	$\rightarrow r_H/\rightarrow r_c$	$\rightarrow r_H/\rightarrow r_c$	$\rightarrow r_H/\rightarrow r_c^{*\dagger}$
+	0	0,-1	-	dS-cosmological singularity	no solution	no solution	no solution	no solution
-	0	0, ± 1	-	dS-naked singularity	$\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c^{*\dagger}$

Table 2

Scale factor and geodesics evolution (charged case).

M	Q^2	k	l^2	$h(R)$	DW	MDB	RDB	Geodesic
+	+	0	+	AdS-naked singularity	$\rightarrow \infty/\text{bounce}$	bounce $/\rightarrow 0$	bounce $/\rightarrow 0$	bounce $^{*\dagger}/\rightarrow \infty^\dagger/\rightarrow 0$
+	+	1	+	AdS-naked singularity	$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow 0$	grow *
+	+	-1	+	AdS-Top.charged black hole	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H$
+	\ll	0,-1	+	AdS-Top.charged black hole	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H^*$
+	\ll	1	+	AdS-Reissner-Nordström	$\rightarrow r_H/\rightarrow \infty$	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H^*$
-	+	0,1	+	AdS-naked singularity	$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow 0$	grow *
-	+	-1	+	AdS-Top.charged black hole	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H$	$\rightarrow r_H^*$
+	+	0,-1	-	dS-naked singularity	$\rightarrow r_c$	$\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c$	$\rightarrow r_c^{*\dagger}$
+	+	1	-	dS-naked singularity	$\rightarrow r_c$	bounce/ $\rightarrow 0/\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c$	$\rightarrow r_c^{*\dagger}$
+	\ll	0,-1	-	dS-naked singularity	$\rightarrow r_c$	$\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c$	$\rightarrow r_c^{*\dagger}/\rightarrow 0$
+	\ll	1	-	dS-Reissner-Nordström	$\rightarrow r_H/\rightarrow r_c$	$\rightarrow r_H/\rightarrow r_c$	$\rightarrow r_H/\rightarrow r_c$	$\rightarrow r_H^*/\rightarrow r_c^{*\dagger}$
-	+	0, ± 1	-	dS-naked singularity	$\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c$	$\rightarrow 0/\rightarrow r_c$	$\rightarrow r_c^{*\dagger}/\rightarrow 0$

Table 3

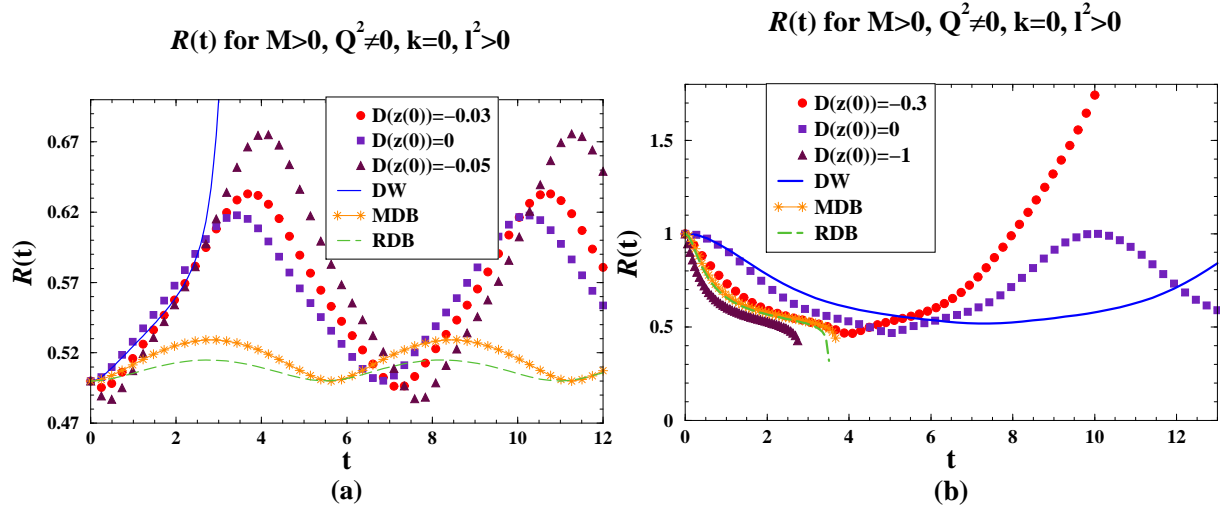
Bulk time t , brane time τ , time delays $\Delta\tau$ and ratio between graviton and photon horizons g/γ for shortcut geodesics.

Fig.	DW				MDB				RDB			
	t	τ	$\Delta\tau$	g/γ	t	τ	$\Delta\tau$	g/γ	t	τ	$\Delta\tau$	g/γ
2a	2.03	.464	.006	1.013	.80	.232	.002	1.008	.64	.192	.001	1.007
2a	-	-	-	-	1.11	.318	.007	1.022	.94	.283	.006	1.020
2b	5.5	1.6	.2	1.206	-	-	-	-	-	-	-	-
2b	6.2	2.5	.1	1.074	-	-	-	-	-	-	-	-

branes getting away from the singularity, falling into it, converging to cosmological horizons when they exist or even bouncing between a minimum and maximum values. The bouncing behavior

is not surprising since in 5 dimensions a similar behavior has been obtained in recent investigations [7], where universes bouncing from a contracting to an expanding phase without encoun-

Figure 2. Scale factor evolution for domain wall, matter and radiation dominated branes and geodesics when $M > 0$, $Q^2 \neq 0$ and $l^2 > 0$ (Anti de Sitter - naked singularity).



tering past and/or future singularities appear. In this way these results could provide support for a singularity-free cosmology or to the so-called cyclic universe scenarios [8].

Finally, we also studied the geodesic behavior. Contrarily to the case of a static brane, where shortcuts appeared under very restrictive conditions [3], the present model of a dynamic brane embedded in a static bulk displays shortcuts in almost all cases and under very mild conditions. Moreover, despite the fact that the time delay between graviton and photon flight time is not percentually so big as in other models [6] (what is also evident from the ratio between graviton and photon horizons), it exists and can eventually be measured by the brane observer, although further considerations are certainly needed in a stricter realistic model. On the other hand, the fact that shortcuts are abundant in the studied setups lends further support to the idea of solving the horizon problem via thermalization by graviton exchange [6,9], however, we should stress that this is not a proof of the solution of the problem yet.

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